Math 122 Wednesday, September 28

$f: G \to G'$ a homomorphism $H = \ker f = \{ g \in G : f(g) = e' \text{ in } G' \}$

easy to see that H is a subgroup: $g, g' \in \ker f \Rightarrow f(gg') = f(g) f(g') = e' e' = e'$
also have inverses $g \in \ker f \Rightarrow f(g^{-1}) = f(g)^{-1} = e'^{-1} = e'$.

Furthermore $H \triangleleft G$ is normal, i.e., H is stable under conjugation
(recall $H \triangleleft G$ iff for all $g \in G$ $h \in H$ $ghg^{-1} \in H$)

$f(ghg^{-1}) = f(g) f(h) f(g^{-1}) = f(g) e' f(g)^{-1} = f(g) f(g)^{-1} = e'$ so $ghg^{-1} \in \ker f$

$F: G \to \mathrm{Aut}(G)$ $g \longmapsto f_g =$ conjugation by g $f_g(x) = gxg^{-1}$

$\ker F = \{ g : f_g = \text{identity on } G \} = \{ g \in G : gxg^{-1} = x \ \forall x \in G \}$
$= \{ g \in G : gx = xg \ \forall x \} = Z$ the <u>center</u> of G

ex $Z(S_n) = \{e\}$ for $n \geq 3$
$Z(G) = G$ whenever G is abelian
$Z(GL_n(\mathbb{R})) = \{ aI : a \in \mathbb{R}^\times \} = \left\{ \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix} \right\}$

Over the next few days we'll show every normal $H \triangleleft G$ is the kernel of some homomorphism $f: G \to G'$

<u>defn</u> On a set S an <u>equivalence relation</u> is a relation $s \sim s'$ that satisfies
1) reflexivity $s \sim s$ $\forall s \in S$
2) symmetry $s \sim s' \Rightarrow s' \sim s$ $\forall s, s' \in S$
3) transitivity $s \sim s', s' \sim s'' \Rightarrow s \sim s''$ $\forall s, s', s'' \in S$

Note: an equivalence relation <u>partitions</u> a set S into disjoint subsets called <u>equivalence classes</u>.

e.g. $\mathbb{Z} = \{\text{even}\} \cup \{\text{odd}\}$
$n \sim m$ iff $n - m$ is even

Often write $T = S/\sim$ = the set of equivalence classes of S

There is a map of sets $S \xrightarrow{h} T$ $s \longmapsto$ equivalence class containing h

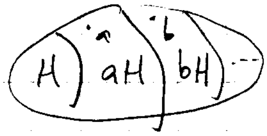
$h^{-1}(h(s)) = \{ s' \sim s \text{ in } S \}$

Now let G be a group and $H \subset G$ be a subgroup. For $a \in G$ the set $aH =$ left coset of a in $G = \{ah : h \in H\}$

Note $aH \leftrightarrow H$ is a bijection of sets with $ah \leftrightarrow h$. This is a bijection because $ah = ah' \Rightarrow a^{-1}ah = a^{-1}ah' \Rightarrow h = h'$.

Note if $a \in H$ then $aH = H$. Clearly $aH \subset H$, and $h = a(a^{-1}h) \in aH$. But if $a \notin H$ then $aH \cap H = \phi$ for if $h \in aH \cap H$ then $h = ah'$, $h' \in H \Rightarrow a = h'h^{-1} \in H$.

This defines an equivalence relation $a \sim b$ iff $aH = bH$ iff $a^{-1}b \in H$. [Check this]
This partitions G:



Note: for this equivalence relation all the equivalence classes (cosets) have the same size.

<u>Notation</u> $T =$ equivalence classes (distinct cosets) $= G/H$.
$[G:H] = \# T = \#$ distinct cosets.

<u>Key Formula</u> $\# G = \# H \cdot [G:H]$ Note each coset has $\#(aH) = \# H$.

<u>Consequences</u>

1) If $\# G$ is finite then $\# H$ divides $\# G$.

2) If $g \in G$ the order of g divides $\# G$. Take $H = \langle g \rangle$.

3) If $\# G$ is a prime p then $G = \langle g \rangle$ for any $g \neq e$ because $1 <$ order of g divides $\# G = p$.

ex $G = \mathbb{Z}$ $a \sim b$ if $a - b$ is even [$a \sim b$ if $a^{-1}b \in H \leftrightarrow b - a$ is even]
$H = 2\mathbb{Z}$ $1 + H = \{odd\}$ $\mathbb{Z}/\sim = \{0, 1\}$

ex $G = \mathbb{Z}$ $H = n\mathbb{Z}$ $n \geq 1$ Cosets are $n\mathbb{Z}, n\mathbb{Z}+1, n\mathbb{Z}+2, \dots, n\mathbb{Z} + (n-1)$
$n = [\mathbb{Z} : n\mathbb{Z}]$ $a \sim b$ iff $a^{-1}b \in n\mathbb{Z}$ i.e. $b - a$ is divisible by n
i.e. $a \equiv b \pmod{n}$ Gauss 1801
$\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ the set of cosets

Now let $f : G \to G'$ a homomorphism and $H = \ker f = eH$

Claim $aH = \{g \in G : f(g) = f(a)\}$ the pre-image of $f(a)$.
Note $g \in aH \Rightarrow g = ah \Rightarrow f(g) = f(ah) = f(a)f(h) = f(a)e' = f(a)$.
If $f(a) = f(b)$ then $f(a^{-1}b) = e' \Rightarrow a^{-1}b \in H$.

In this case $T = G/H$ = Image of $f \subset G'$ and $\#G = \underbrace{\# \ker f}_{\#H} \times \underbrace{\# \operatorname{im} f}_{[G:H]}$

Note G/H is a priori a set but here it is $\operatorname{Im} f$ a group!

In general, if $H \triangleleft G$, we will put a group structure on G/H [next time]

ex. $S_n \xrightarrow[\text{sign}]{} \{\pm 1\}$ $\quad p \longmapsto \det(A_p)$ $\quad \operatorname{Im}(\text{sign}) = \pm 1$ because for $n \geq 2$

$$A_{(12)} = \begin{pmatrix} 0 & 1 & & 0 \\ 1 & 0 & & \\ & & 1 & \\ 0 & & & \ddots & 1 \end{pmatrix}$$ s.t. $\det A_{(12)} = -1$. $\quad \ker(\text{sign}) = A_n$ = even permutations

Know $\# S_n = \# A_n \cdot \#\text{image} = \# A_n \times 2 \Rightarrow \# A_n = \frac{n!}{2}$

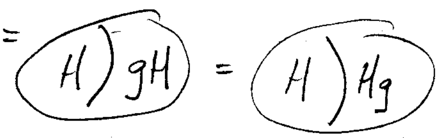
the coset $(12) A_n = \{\text{odd permutations}\}$ and also has order $\frac{n!}{2}$

Claim $H \subset G$ index $= [G:H] = 2$ then $H \triangleleft G$

[Note if $[G:H] = 1$ then $G = H$ is certainly normal.]

[Aside aH = left coset of a, Ha = right coset of a but no reason why aH must equal Ha]

If $H \triangleleft G$ then $gHg^{-1} = H$, i.e., $gH = Hg$ $\forall g \in G$.

Proof by picture: $G =$ 

$g \notin H$ so $gH = Hg$